Surface Operators in 4d/2d Duality
-on TQFT picture of 4d N=2 Argyres-Douglas Theories

Rui-Dong Zhu
Dublin Institute for Advanced Studies

collaboration with T. Nishinaka (Ritsumekan U.), S. Sasa (U. Tokyo)

and A. Watanabe (U. Tokyo), A. Tsuchiya (IPMU)

based on arXiv: 1811.11772 (published on JHEP)

and work in progress

PLAN of TALK

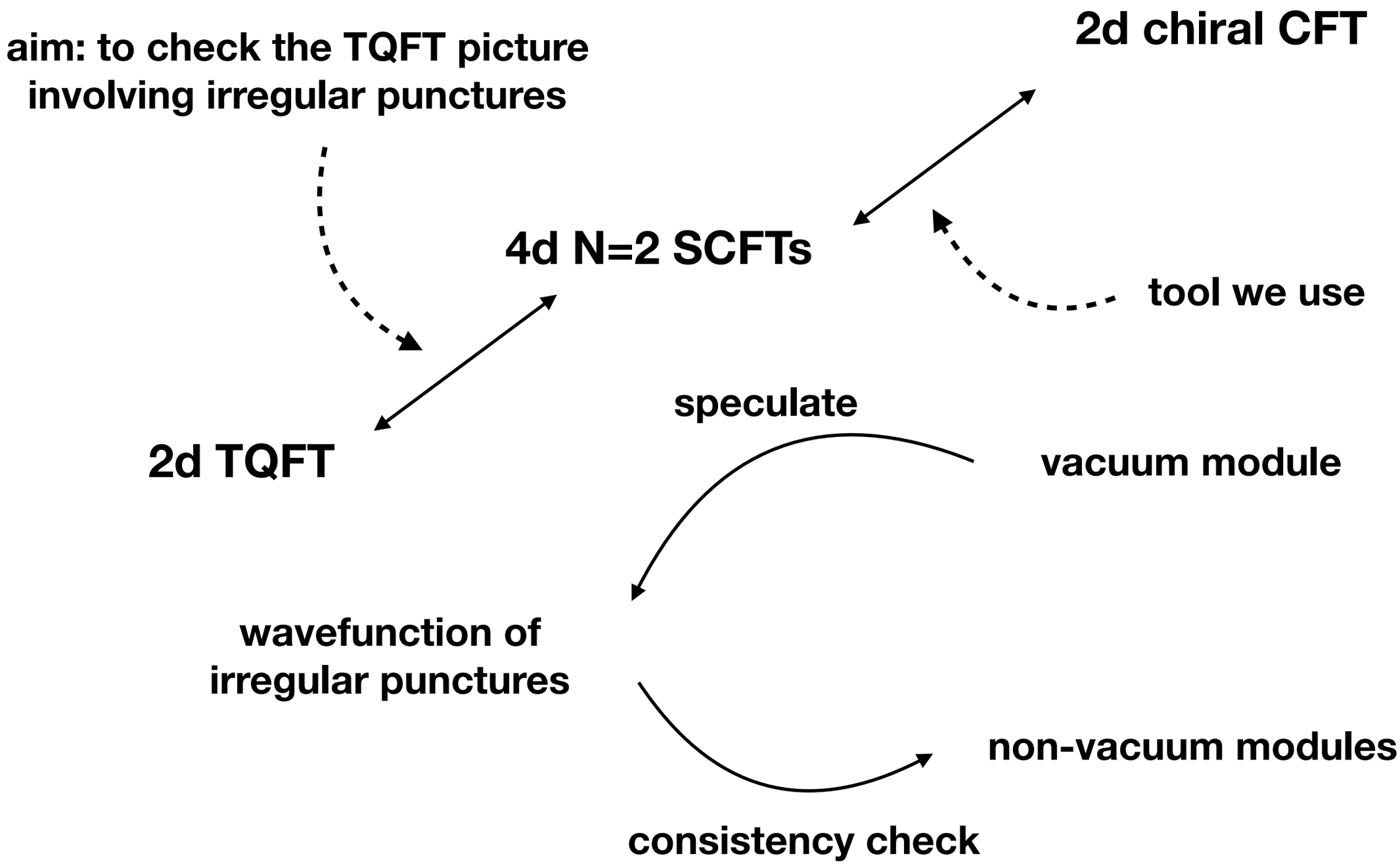
1. Introduction and Motivation

2. A Brief Review on Argyres-Douglas (AD) Theories

3. Higgsing Method and Surface Operators

4. Surface Operators in AD vs Modules in Chiral Algebra

What we did:


aim: to check the TQFT picture
involving irregular punctures
2d chiral CFT
4d N=2 SCFTs
tool we use
2d TQFT
speculate
vacuum module
wavefunction of
irregular punctures
non-vacuum modules
consistency check

- **It is well known that a very special theory, 6d N=(2,0) theory, arises as the low energy effective theory of M5 branes.**

[Witten (1995)], [Schwarz (1995)] …

- **This theory is notorious for having no Lagrangian description, due to the self-dual 2-form field contained in it.**

- **Yet we know its field contents, and thus we know what we will obtain when we compactify this theory to lower dimensional (gauge) theories.**

- **Considering such compactifications provides detours to study these 6d theories, but on the hand, they also predict a lot of unexpected properties for supersymmetric gauge theories in lower dimensions.**

Consider computing some topological quantity of 6d N=(2,0) theory

on $M_d \times M_{6-d}$

zero size M_{6-d} | **zero size M_d**

(topological) physical quantities in some theory (labeled by M_{6-d}) on M_d = **(topological) physical quantities in some theory on M_{6-d}**

There might not be any interesting physical quantities appearing in this setup, but…

For d=4 **and M_{6-d} = punctured Riemann surface**

on the 4d side

we can preserve half of the supercharges during the compactification

[Gaiotto-Witten (2008)]

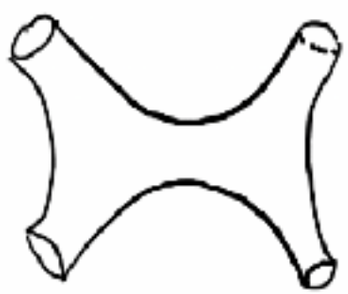
and resulting large class of 4d theories are usually called class S theories.

Examples for SU(2) gauge group

(with only regular punctures)

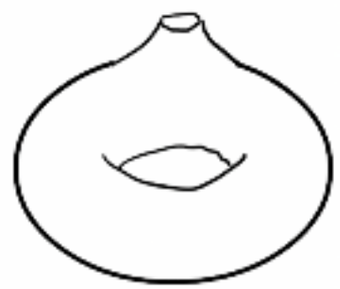
sphere with three punctures

(three) free hypermultiplets



sphere with four punctures

SU(2) N_f=4



torus with one puncture

N=2* theory

(with adjoint hypermultiplet)

4d/2d duality

6d N=(2,0) Theory
(M5 branes)

[Alday-Gaiotto-Tachikawa (2009)]
[Wyllard (2009)]
[Gadde-Rastelli-Razamat-Yan (2011)]
[Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees (2013)]
...

Compactify on a punctured Riemann surface

4d N=2 Gauge Theory with gauge group *G*

⟷

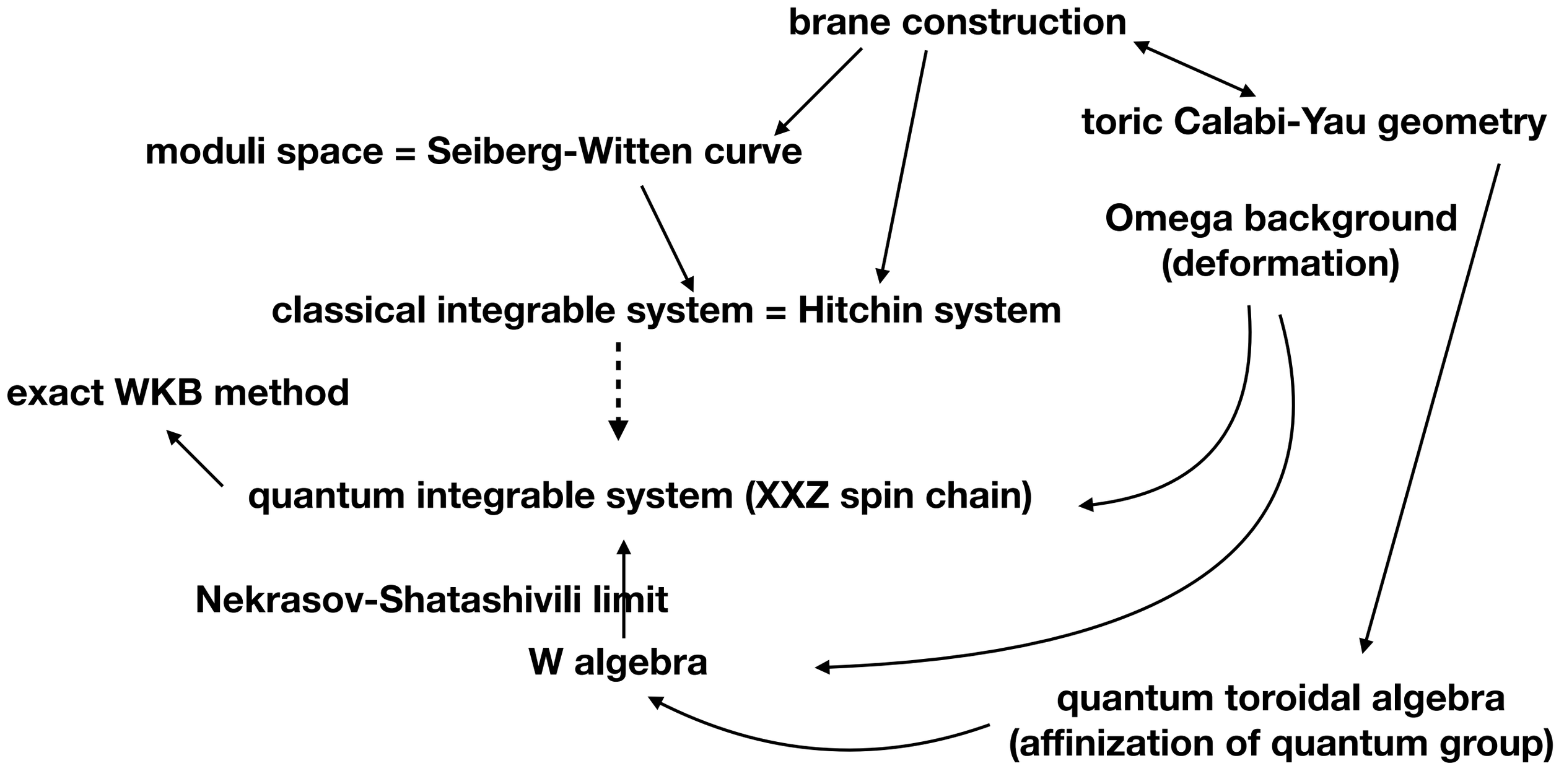
(certain) 2d Theory living on the Riemann surface

(Class S)

- S^4 partition function = correlation function of 2d CFT with W_G symmetry on the Riemann surface.
- $S^3 \times S^1$ partition function (superconformal index) = correlator in 2d TQFT (q-YM theory)

- **S^4 partition function = correlation function of 2d CFT with W_G symmetry on the Riemann surface.**

It is also an interesting topic.



4d/2d duality

6d N=(2,0) Theory
(M5 branes)

[Alday-Gaiotto-Tachikawa (2009)]
[Wyllard (2009)]
[Gadde-Rastelli-Razamat-Yan (2011)]
[Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees (2013)]
...

Compactify on a punctured Riemann surface

4d N=2 Gauge Theory with gauge group *G*

(Class S)

⟷

(certain) 2d Theory living on the Riemann surface

- S^4 partition function = correlation function of 2d CFT with W_G symmetry on the Riemann surface.
- $S^3 \times S^1$ partition function (superconformal index) = correlator in 2d TQFT (q-YM theory)

$S^3 \times S^1$ partition function = Witten index

[Aharony-Marsano-Minwalla-Papadodimas-Van Raamsdonk (2003)]

[Kinney-Maldacena-Minwalla-Raju (2005)]

$$\mathcal{I}(p,q,t) = \mathrm{tr}\left[(-1)^F p^{\frac{E-2j_1-2R-r}{2}} q^{\frac{E+2j_1-2R-r}{2}} t^{R+r} e^{-\beta H}\right],$$

E: conformal weight $j_{1,2}$: spin R: $SU(2)_R$ charge r: $U(1)_r$ charge

Witten index with respect to "Hamiltonian" H=E-2j_2-2R+r

$$H = 2\left\{\bar{\mathcal{Q}}_{1\dot{-}}, \bar{\mathcal{Q}}^{\dagger}_{1\dot{-}}\right\} = 2(E - 2j_2 - 2R + r).$$

counting shorten (BPS) multiplets

- **Given the Lagrangian of a theory, we can (analytically) compute the partition function exactly (with the technology of localization).**

[Aharony-Marsano-Minwalla-Papadodimas-Van Raamsdonk (2003)]

[Festuccia-Seiberg (2011)]

[Closset-Shamir (2013)]

[Assel-Cassani-Martelli (2014)]

Matrix Model Expression of Superconformal index

[Aharony-Marsano-Minwalla-Papadodimas-Van Raamsdonk (2003)]

$$\mathcal{I}(p,q,t;V) = \oint [\mathrm{d}U] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{R_j}(p^n,q^n,t^n)\chi_{R_j}(U^n,V^n)\right).$$

element of flavor symmetry group

Haar measure

field contents in rep. R_j of the gauge group and flavor symmetry

character of representation R_j

N=2 theory:

vector multiplet and hypermultiplet

- [Dolan-Osborn (2008)]

  indices all written in terms of elliptic Gamma function

  hypermultiplet in fundmental rep. of SU(n) $\displaystyle\prod_{i=1}^{n} \Gamma\left(\frac{t}{(pq)^{1/3}} a_i^{\pm}; p, q\right)$

  vector multiplet $\displaystyle\frac{\left[\Gamma\left(\frac{pq}{t};p,q\right)(p;p)(q;q)\right]^{n-1}}{\Delta(\mathbf{a})\Delta(\mathbf{a}^{-1})} \prod_{i\neq j} \frac{\Gamma\left(\frac{pq}{t}a_i/a_j;p,q\right)}{\Gamma\left(a_i/a_j;p,q\right)}$

  Haar measure

- **[Gadde-Rastelli-Razamat-Yan (2011)]**

$S^3 x S^1$ partition function (superconformal index) = correlator in TQFT

TQFT = q-deformed Yang-Mills theory

considered

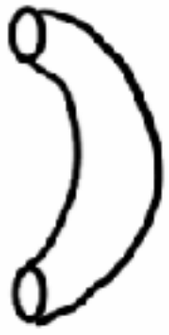
[Buffenoir-Roche (1995)]
[Klimcik (1999)]) **quantum group**

[Aganagic-Ooguri-Saulina-Vafa (2005)]

BH vs topological string

TQFT $\Longrightarrow$ pants decomposition of Riemann surface

All we need to specify to compute the partition function



cap (to close a puncture)

- **[Gadde-Rastelli-Razamat-Yan (2011)]**

$S^3 x S^1$ partition function (superconformal index) = correlator in TQFT

TQFT = q-deformed Yang-Mills theory

considered

[Buffenoir-Roche (1995)]
[Klimcik (1999)]) **quantum group**

[Aganagic-Ooguri-Saulina-Vafa (2005)]

BH vs topological string

TQFT $\Longrightarrow$ pants decomposition of Riemann surface

All we need to specify to compute the partition function

$$\Gamma\left(\frac{t}{(pq)^{1/3}}a^{\pm}b^{\pm}c^{\pm};p,q\right)$$

What is more surprising and convenient for computation:

[Gadde-Rastelli-Razamat-Yan (2011)]

There must exist a proper basis, orthonormal, that diagonalizes the three point function.

puncture = state in q-YM

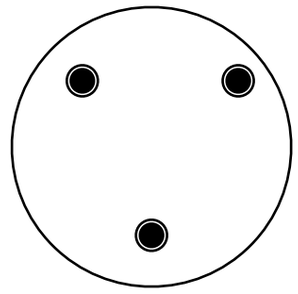
$$\mathcal{I}(p,q,t;x_k^{(i)}) = \sum_R C_R^{2g-2+m}(p,q,t)\prod_{i=1}^{m} f_R^{(i)}(p,q,t;x_k^{(i)}),$$

where C_R is the structure constant of q-YM, and f_R is the wave function of the corresponding state.

g: genus of the Riemann surface
m: number of punctures on the Riemann surface
R: Young diagram label of the Hilbert space

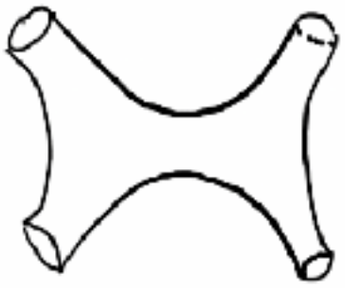
e.g. sphere with three punctures

$$\mathcal{I}(p,q,t;x_k^{(i)}) = \sum_R C_R(p,q,t) \prod_{i=1}^{3} f_R^{(i)}(p,q,t;x_k^{(i)}).$$



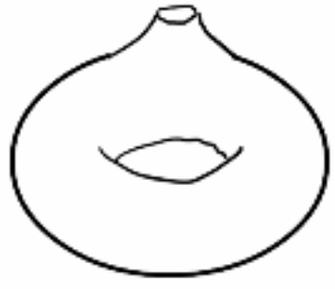
e.g. sphere with four punctures

$$\mathcal{I}(p,q,t;x_k^{(i)}) = \sum_R C_R^2(p,q,t) \prod_{i=1}^{4} f_R^{(i)}(p,q,t;x_k^{(i)}).$$



e.g. torus with one puncture

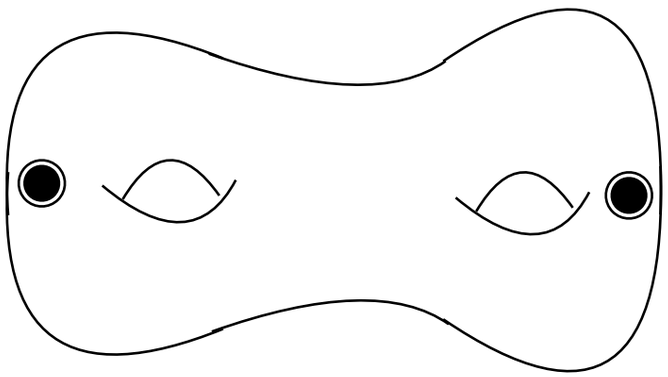
$$\mathcal{I}(p,q,t;x_k^{(1)}) = \sum_R C_R(p,q,t) f_R^{(1)}(p,q,t;x_k^{(1)}).$$



It would be done if we had a simple expression for C_R and f_R!

What kind of punctures can be considered in class S theories?

- **Regular puncture, characterized by the b.c.**



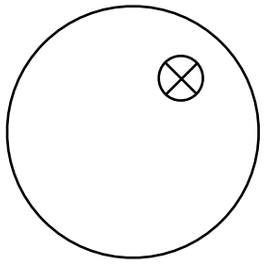
$$\Phi = \frac{T}{z} + \dots$$

(Gaiotto curve)

“basic” ingredient

there is an G=SU(N) flavor symmetry attached to the full regular puncture.

- **Irregular punctures**



$$\Phi = \frac{T}{z^{2+k/b}} + \cdots .$$

b.c. not a pole

  - **can be obtained by colliding several regular punctures together**

[Xie (2012)]

[Xie-Wang (2015)]

Macdonald limit ($p \to 0$)

[Gadde-Rastelli-Razamat-Yan (2013)]

In this limit, one can find a convenient and orthogonal basis (spanned by normalized Macdonald polynomials) to diagonalize the three-point structure of the q-YM theory. We have a very clean expression for the superconformal index.

three-point structure:

normalized Macdonald polynomial

$$C_\lambda^{-1} = \frac{\bar{P}_\lambda(t^\rho)}{\prod_{i=1}^{r}(t^{d_i};q)_\infty},$$

Young diagram

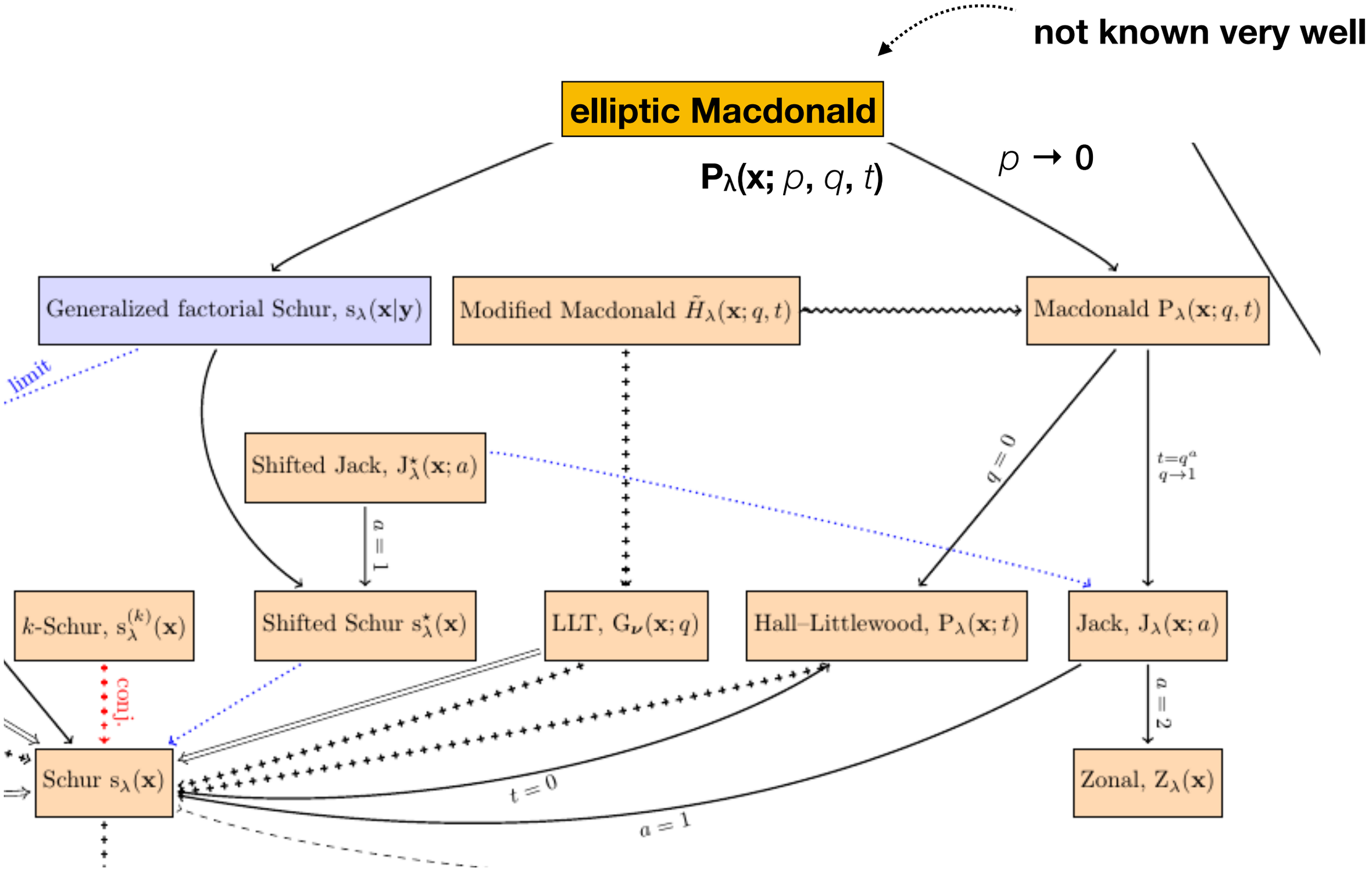
degree of Casimir

wave function for a (full) regular puncture:

$$f_\lambda(x) = \frac{\bar{P}_\lambda(x)}{(t;q)^r \prod_{\alpha\in\Delta}(tx^\alpha;q)}$$

set of roots

Story of symmetric Polynomials



How do we get the expressions for wavefunctions of irregular punctures?

How will we check if we have a proposal?

Duality between 4d N=2 SCFT and 2d chiral CFT

In the limit p → 0 (Macdonald limit), only Schur operators contribute to the index.

Schur operator:

$$E = j_1 + j_2 + 2R, \quad r = j_2 - j_1.$$

- **[Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees (2013)]**

4d N=2 SCFT ⟷ **2d chiral algebra**

living on x^3-x^4 plane (pt x R^2 in R^4)

Schur index = character of 2d chiral algebra

↑

Schur limit, p → 0, q = t, of the superconformal index

In particular [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees (2013)]

R-symmetry current $\subset$ **stress tensor multiplet**

$\longrightarrow$ **Virasoro operator** $c_{2d} = -12 c_{4d}.$

non-unitary CFT!

conformal weight $h = R + j_1 + j_2$

Schur index

$$\mathcal{I}(q) = \mathrm{tr}\left[(-1)^F q^{R+j_1+j_2}\right] = \mathrm{tr}\left[(-1)^F q^h\right],$$

gives the vacuum character of the corresponding 2d chiral algebra.

flavor symmetry $\longrightarrow$ **affine Kac-Moody symmetry**

$$k_{2d} = -\frac{k_{4d}}{2}$$

In particular [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees (2013)]

R-symmetry current $\subset$ **stress tensor multiplet**

$\longrightarrow$ **Virasoro operator** $c_{2d} = -12 c_{4d}.$

non-unitary CFT!

conformal weight $h = R + j_1 + j_2$

Schur index

$$\mathcal{I}(q) = \mathrm{tr}\left[(-1)^F q^{R+j_1+j_2}\right] = \mathrm{tr}\left[(-1)^F q^h\right],$$

gives the vacuum character of the corresponding 2d chiral algebra.

flavor symmetry $\longrightarrow$ **affine Kac-Moody symmetry**

$$k_{2d} = -\frac{k_{4d}}{2}$$

How are non-vacuum modules of the chiral algebra realized?

recall that the conformal weight of local operators takes the form,

$$h=R+j_1+j_2 \in \frac{1}{2}\mathbb{Z}$$

a very limited set of operators in 2d chiral algebra can be realized.

Moreover, from the discussion of saturation of unitarity bound (in 4d), we can see that Schur operators are always highest-weight states of $SU(2)_R$ and Lorentz group.

i.e.

$$h \geq 0$$

and h=0 only for the identity operator.

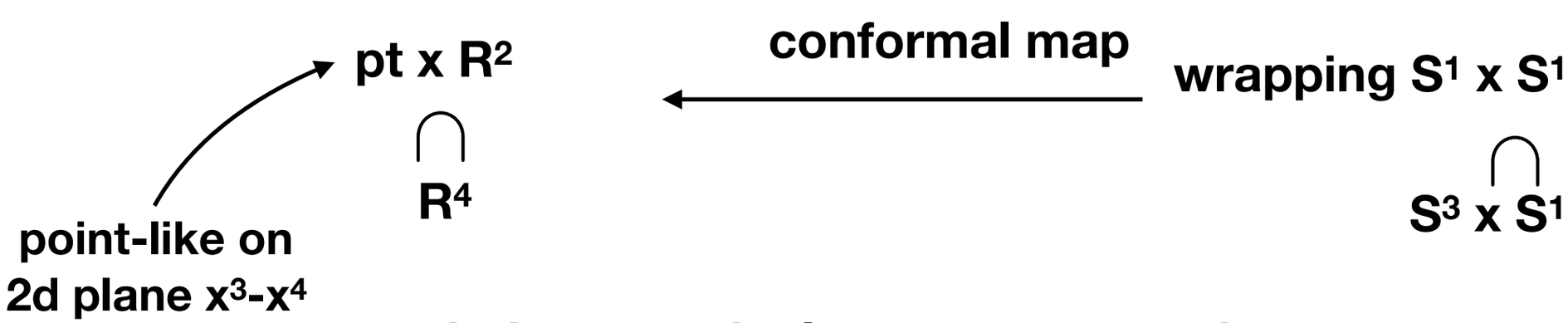
How are non-vacuum modules of the chiral algebra realized?

Studied in several simple examples of Argyres-Douglas theories.

- **line operator** **[Cordova-Gaiotto-Shao (2016)]**

index = linear combination of non-vacuum characters

- **surface operator** **[Cordova-Gaiotto-Shao I, II (2017)]**

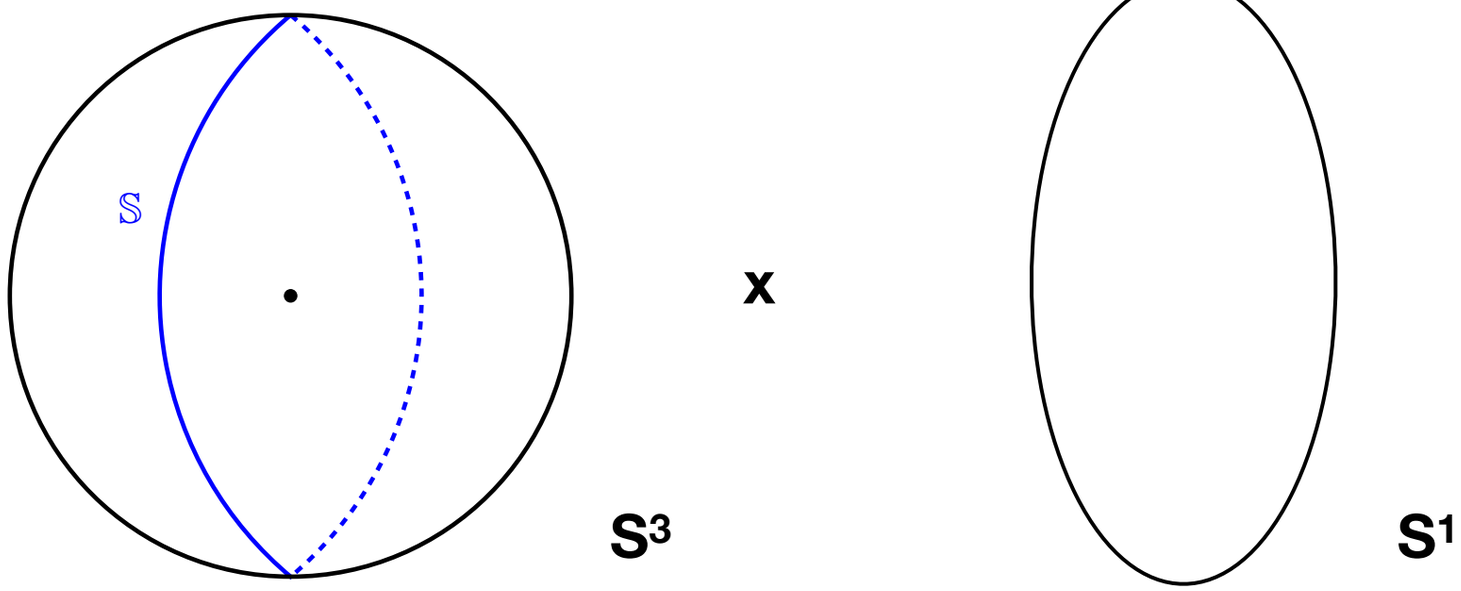


index = particular non-vacuum character

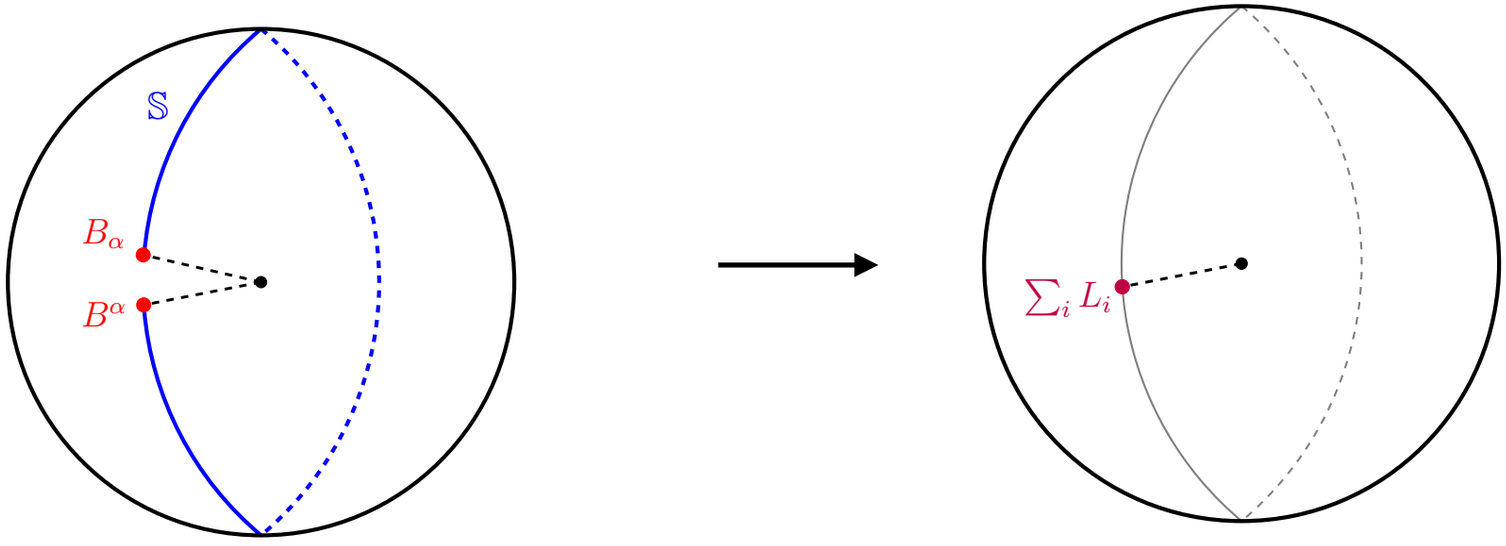
decompose surface operator into line operators

setup of surface operator

[Cordova-Gaiotto-Shao (2017)]



by inserting a pair of operators that form a complete basis



How to realize non-vacuum modules of the chiral algebra?

Studied in several simple examples of Argyres-Douglas theories.

- **line operator** [Cordova-Gaiotto-Shao (2016)]

ray starting from the origin $\xleftarrow{\text{conformal map}}$ wrapping pt x S^1

$\cap$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cap$

R^4 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $S^3 \times S^1$

index = linear combination of non-vacuum characters

- **surface operator** [Cordova-Gaiotto-Shao I, II (2017)]

pt x R^2 $\xleftarrow{\text{conformal map}}$ wrapping $S^1 \times S^1$

$\cap$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cap$

R^4 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $S^3 \times S^1$

point-like on 2d plane x^3-x^4

index = particular non-vacuum character

What to check?

Argyres-Douglas (AD) theories ~ “Simplest” SCFTs

- **first discovered as a non-trivial IR fixed point at special point of Coulomb branch**

[Argyres-Douglas (1995)]

- **Coulomb branch operators have generally non-integer conformal dimensions**

- **No Lagrangian descriptions (no marginal deformation associated to couplings)**

- **“smallest” central charge c in N=2 theories**

[Liendo-Ramirez-Seo (2015)]

$$c \geq \frac{11}{30}$$

- **constructed from sphere with one irregular puncture (potentially with one more regular puncture)**

[Xie (2012)]

[Xie-Wang (2015)]

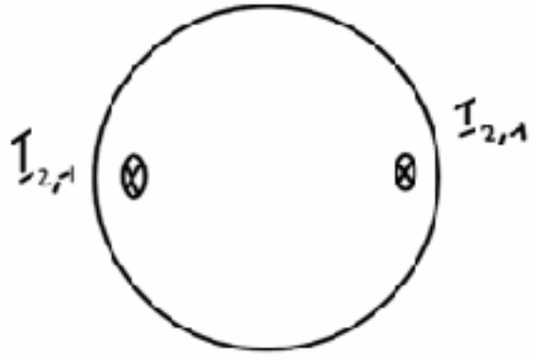
$$\Phi = \frac{T}{z^{2+k/b}} + \cdots .$$

Wavefunctions for irregular punctures

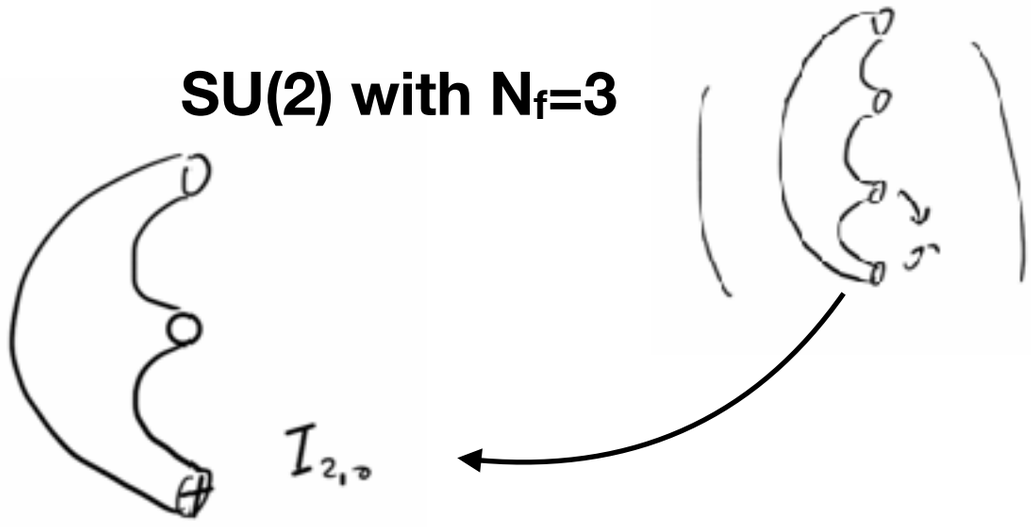
[Song (2015)]

lessons from Lagrangian theories

ex. pure N=2 SU(2) Yang-Mills



SU(2) with N_f=3



- **$I_{2,-1}$ puncture = gauged regular puncture**

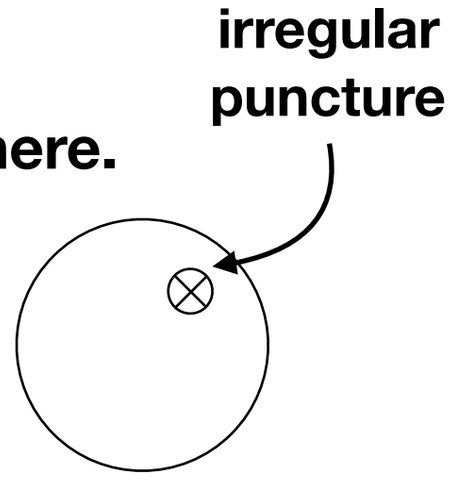
$$f_\lambda^{I_{2,-1}}(q,t) = \oint \frac{\mathrm{d}z}{2\pi i z} I_{vec}(z) f_\lambda(z;q,t).$$

- **$I_{2,0}$ puncture = gauged regular puncture with hypermultiplet attached**

$$f_\lambda^{I_{2,0}}(x;q,t) = \oint \frac{\mathrm{d}z}{2\pi i z} I_{vec}(z;q,t) I_{hyp}(z,x;q,t) f_\lambda(z;q,t).$$

Argyres-Douglas (AD) theories ~ "Simplest" SCFTs

In class S picture, constructed with an irregular puncture on a sphere.



We consider the following two classes of AD theories:

1) (A_{N-1}, A_{M-1}) theories with gcd(N,M)=1

- **dual to (N,N+M) minimal model CFTs.**

[Cordova-Shao (2015)]

- **they are constructed from the irregular puncture $I_{N,M}$, and the wavefunction is conjectured to be given by the rescaling**

$$f_\lambda^{I_{N,M}}(q) = f_\lambda^{I_{N,1-N}}(q^{N+M}).$$

[Song (2015)]

both sides of the 4d/2d duality are well-known (studied).

2) (A_{N-1}, $A_{N(k-1)-1}$) theories i.e. gcd(N,N(k-1))=N

[Buican-Nishinaka (2015)]
[Buican-Nishinaka (2017)]

- **the wave function is also conjectured as**

"free boson" not rescaled

$$f_\lambda^{I_{N,N(k-1)}}(q) = \left(\frac{(q^k;q^k)_\infty}{(q;q)_\infty}\right)^{N-1} f_\lambda^{I_{N,0}}(q^k).$$

2) (A_{N-1}, $A_{N(k-1)-1}$) theories i.e. gcd(N,N(k-1))=N

flavor symmetry: $U(1)^{N-1}$ ⟶ higher-rank free boson

the complete dual chiral algebra: rank N-1 free boson $\rtimes$ W_N algebra

central charge

$$c = 2(N-1) - \frac{\rho^2 (k-1)^2}{2k}.$$

(free boson) N-1 + W_N algebra with $c_{N,k} = (N-1) - \frac{\rho^2 (k-1)^2}{2k}.$

One of the best way to study this chiral algebra is the free field realization (VOA approach).

...
[Feigin et al. (2006)]
[Feigin-Tipunin (2010)]
[Tsuchiya-Wood (2013)]
[Creutzig (2017)]
[Creutzig (2018)]
...

These conjectures

$$f_\lambda^{I_{N,M}}(q) = f_\lambda^{I_{N,1-N}}(q^{N+M}).$$

$$f_\lambda^{I_{N,N(k-1)}}(q) = \left(\frac{(q^k;q^k)_\infty}{(q;q)_\infty}\right)^{N-1} f_\lambda^{I_{N,0}}(q^k).$$

are more or less based on the evidence that they reproduce the vacuum character of the dual chiral algebra. We do not have a clear physical picture as

$$f_\lambda^{I_{2,-1}}(q,t) = \oint \frac{\mathrm{d}z}{2\pi i z} I_{vec}(z) f_\lambda(z;q,t).$$

$$f_\lambda^{I_{2,0}}(x;q,t) = \oint \frac{\mathrm{d}z}{2\pi i z} I_{vec}(z;q,t) I_{hyp}(z,x;q,t) f_\lambda(z;q,t).$$

We would like to further check these conjectures against the correspondence between surface operators and non-vacuum modules of chiral algebra.

How to compute the index with surface operator?

- **localization**

  famous results when there exists Lagrangian description

- **BPS spectra**

  Using Kontsevich-Soibelman (KS) operators

  method used in [Cordova-Gaiotto-Shao (2017)]

- **Higgsing method** **[Gaiotto-Rastelli-Razamat (2012)]**

  Most systematic and does not need Lagrangian description.

  In the class S picture, this corresponds to insert a regular puncture to the original theory, and to give a position-depending VEV to the flavor symmetry associated to the new puncture.

How to compute the index with surface operator?

- **localization**

  famous results when there exists Lagrangian description

- **BPS spectra**

  Using Kontsevich-Soibelman (KS) operators

  method used in [Cordova-Gaiotto-Shao (2017)]

- **Higgsing method** **[Gaiotto-Rastelli-Razamat (2012)]**

  Most systematic and does not need Lagrangian description.

  In the class S picture, this corresponds to insert a regular puncture to the original theory, and to give a position-depending VEV to the flavor symmetry associated to the new puncture.

Higgsing method in class S picture

[Gaiotto-Rastelli-Razamat (2012)]

1. prepare a UV theory by inserting an additional full regular puncture to the original theory (constructed from the 6d N=(2,0) theory of type A_{N-1})

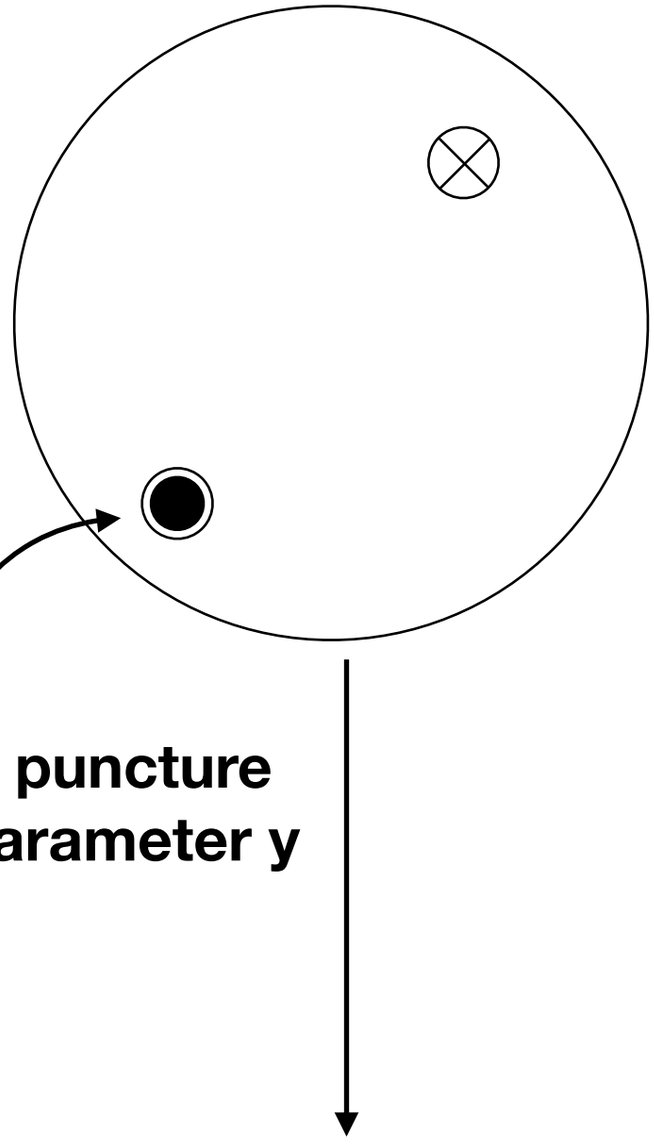
2. there will be an extra SU(N) flavor attached to the additional full regular puncture, and one can consider a position-dependent VEV for this flavor symmetry (by Higgsing the momentum map operator)

3. we flow back to the IR and neglect decoupled d.o.f's , resulting in the original theory with a surface operator inserted.

Schematically,

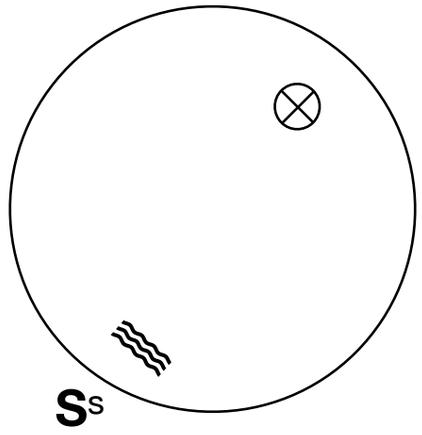
I_{UV}

$y \to y^*$



I_{IR} **with surface operator** S^s

decoupled d.o.f. **x**



As for the Schur index

taking the fugacity to a particular value (corresponding to the VEV), and then decoupling the Nambu-Goldstone modes during the breaking of the flavor symmetry G=SU(N).

simple roots

$$\mathcal{I}_{\rm IR}(q;\mathbf{x}) = \Big(\mathcal{I}_{\rm vec}(q)\Big)^{\frac{N(N-1)}{2}} \cdot \lim_{\mathbf{y}\to\mathbf{y}^*} \left(\mathcal{I}_{\rm UV}(q;\mathbf{x},\mathbf{y}) \prod_{1\le i\le j\le N-1} \left(1-\prod_{k=i}^{j}(q y^{\alpha_k})\right)\right) \ ,$$

where y* is the solution to $q^{1+s_\ell} y^{\alpha_\ell} = 1, \quad \text{for} \qquad \ell = 1,\cdots,N-1$**.**

s_ℓ **labels the corresponding surface operator obtained in the IR theory.**

$\longleftrightarrow$ **giving VEV to operator** $(\sigma^\mu_{++}\partial_\mu)^{s_\ell}\mathcal{O}_{\alpha_\ell}$ **.**

momentum map

y*'s reproduce the poles in the wave function of the regular puncture:

[Gadde-Rastelli-Razamat-Yan (2011)]

$$f_R^{\rm full}(q,\mathbf{x}) = \frac{\chi_R^{\rm SU(N)}(\mathbf{x})}{(q;q)^r \prod_{\alpha\in\Delta}(q x^\alpha;q)}$$

Given the expression of the Schur index, we want to investigate the correspondence between the label of the surface operators and the non-vacuum modules in 2d chiral algebra.

decoupling of hypermultiplets

$$I_{hyp}=(yt^{1/2};q)_\infty^{-1}(y^{-1}t^{1/2};q)_\infty^{-1}$$

with U(1) flavor symmetry

$\downarrow$ $\quad y\to t^{1/2}q^s$ **for some positive integer s**

$$\underbrace{(1-t^{1/2}q^s/y)^{-1}}_{\text{“massless” mode}}\quad\underbrace{(t;q)_\infty^{-1}(q;q)_\infty^{-1}}_{I_{vec}}\ x\ \underbrace{\frac{(1-t)(1-tq)\dots(1-tq^{s-1})}{(1-q^{-1})(1-q^{-2})\dots(1-q^{-s})}}_{\text{strip-off factor}}$$

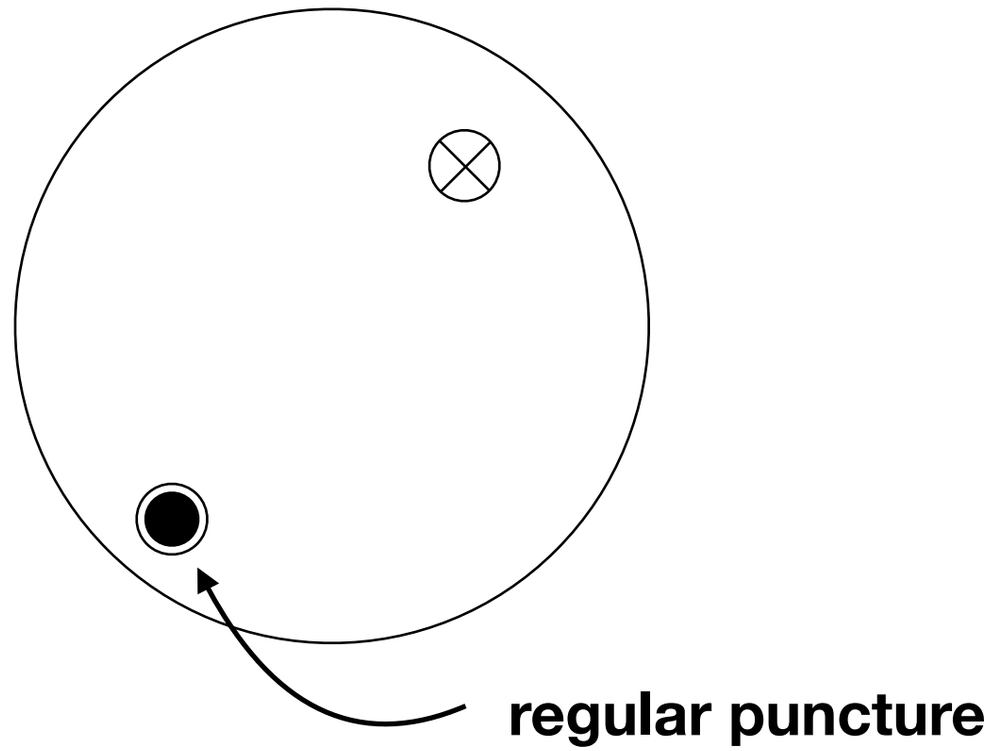
trivializes in the Schur limit (q=t)

For SU(N) flavor symmetry, we will have |Δ| such factors from decoupled hypermultiplets.

index with surface operator

I_{UV} =

$y \to y^*$ satisfying $q^{1+s_\ell} y^{\alpha_\ell} = 1,$



I_{IR} with surface operator S^s

compare the index with surface operator to the character of 2d chiral algebra.

For convenience, we define

$$n_\ell \equiv 1 + s_\ell \ .$$

Results I:

1) W_N minimal models (P=N, Q=N+M)

with level rank duality corresponding to exchanging (A_{N-1}, A_{M-1})

finite number of non-singular modules labeled by N positive integers

$\vec{n} = (n_i)_{i=1,\ldots,N}$ **satisfying** $\sum_{i=1}^{N} n_i = Q,$

the well-known character formula:

$$\chi_{\vec{n}}^{(N,N+M)}(q) = \frac{(q^{N+M};q^{N+M})^{N-1} \prod_{i,j}(q^{a_{i,j}};q^{N+M}) \prod_{i,j}(q^{N+M-a_{i,j}};q^{N+M})}{(q;q)^{N-1}},$$

where $a_{i,j} = n_j + n_{j+1} + \cdots + n_{j+i-1}, \text{ for } i,j \in \mathbb{N} \text{ such that } i+j \leq N.$

correspondence with surface operators:

$$n_\ell \equiv 1 + s_\ell \qquad \longleftrightarrow \qquad \vec{n} = (n_i)_{i=1,\ldots,N}$$

- **We observe permutation symmetry among n_i's and periodicity of n_i's in the index expression.**

Results II:

2) (A_1, A_{2k-3}) theories

2d dual: $\mathcal{B}_k$ **algebra**

modules:

$$W_s = \bigoplus_{r\in\mathbb{Z}} M_{r+1,s} \otimes \mathcal{F}_{\sqrt{-k}r} \ ,$$

Fock space

direct sum of Virasoro modules

$$M_{r,s} \equiv \bigoplus_{\ell=0}^{\infty} L(h_{r+2\ell,s},\, c_k) \ \text{for} \ r \geq 1$$

$$M_{r,s} \equiv \bigoplus_{\ell=0}^{\infty} L(h_{r-1-2\ell,k-s},\, c_k) \ \text{for} \ r \leq 0$$

- **known character** ✔

  correspondence rule:

  s ⟷ **n**

  As a natural generalization of the work on the vacuum module of the algebra.

[Creutzig (2015)]

Results III:

3) (A_{N-1}, $A_{N(k-1)-1}$) theories

generalization of the 2d chiral algebra: $\mathcal{B}(k)_{A_{N-1}}$

our conjecture:

modules of $\mathcal{B}(k)_{A_{N-1}}$

$$W_\nu = \bigoplus_{\mu \in P} \mathcal{M}(k,\mu,\nu) \otimes \mathcal{F}_{\sqrt{-k}\mu}$$

ν : Young diagram characterized by n_i's and corresponding to representation of SU(N)

character of $\mathcal{M}(k,\mu,\nu)$

$$\chi(k,\mu,\nu) \equiv \frac{q^{\frac{\nu^2}{2k}}}{\eta(q)^{N-1}} \sum_{\lambda \in P^+ \cap (Q+\mu)} q^{\frac{k}{2}(\lambda+\rho)^2} m_\lambda(\mu) \sum_{w \in W} \epsilon(w) q^{-(\nu, w(\lambda+\rho))}$$

multiplicity of μ in rep. λ

Evidence:

- **worked perfectly in the case N=2**
- **agrees with the known vacuum result for N>2**

[Creutzig (2017)]

Towards generalization:

The character formula

$$\chi(k,\mu,\nu) \equiv \frac{q^{\frac{\nu^2}{2k}}}{\eta(q)^{N-1}} \sum_{\lambda \in P^+ \cap (Q+\mu)} q^{\frac{k}{2}(\lambda+\rho)^2} m_\lambda(\mu) \sum_{w \in W} \epsilon(w) q^{-(\nu, w(\lambda+\rho))}$$

reflects the singular vector structure in the module.

For example in the case of A_1, we have the following exact sequence,

$$\cdots \to \mathcal{F}_{\lambda_{r-2,s}} \overset{\mathcal{Q}_-^{[s]}}{\to} \mathcal{F}_{\lambda_{r-1,p-s}} \overset{\mathcal{Q}_-^{[p-s]}}{\to} \mathcal{F}_{\lambda_{r,s}} \overset{\mathcal{Q}_-^{[s]}}{\to} \mathcal{F}_{\lambda_{r+1,p-s}} \to \cdots$$

Thus modules defined as the kernel of screening charges

$$M_{r,s} = \mathrm{Ker}\mathcal{Q}_-^{[s]} : \ \mathcal{F}_{\lambda_{r,s}} \to \mathcal{F}_{\lambda_{r,-s}}$$

have the character of the form

$$\chi(M_{r,s}) = \sum_{n \geq 0} \left(\chi(\mathcal{F}_{\lambda_{r-2n-1,p-s}}) - \chi(\mathcal{F}_{\lambda_{r-2n-2,s}}) \right).$$

Our conjectured formula takes a similar form and one can check it in some simple cases of A_2.

[Tsuchiya-RZ, work in progress]

The vacuum module of $\mathcal{B}(k)_{A_{N-1}}$ is a common kernel of N-1 screening charges.

$$\bigcap_{i=1}^{N-1} \mathrm{Ker}\mathcal{Q}_{-}^{(i)[1]}$$

(over some direct sum of Fock spaces)

c.f. rank one case

$$W_s : \ \mathrm{Ker}\mathcal{Q}_{-}^{[s]}$$

conjecture: in higher-rank case

module is labeled by a Young diagram $\nu = \{\nu_i\}$,

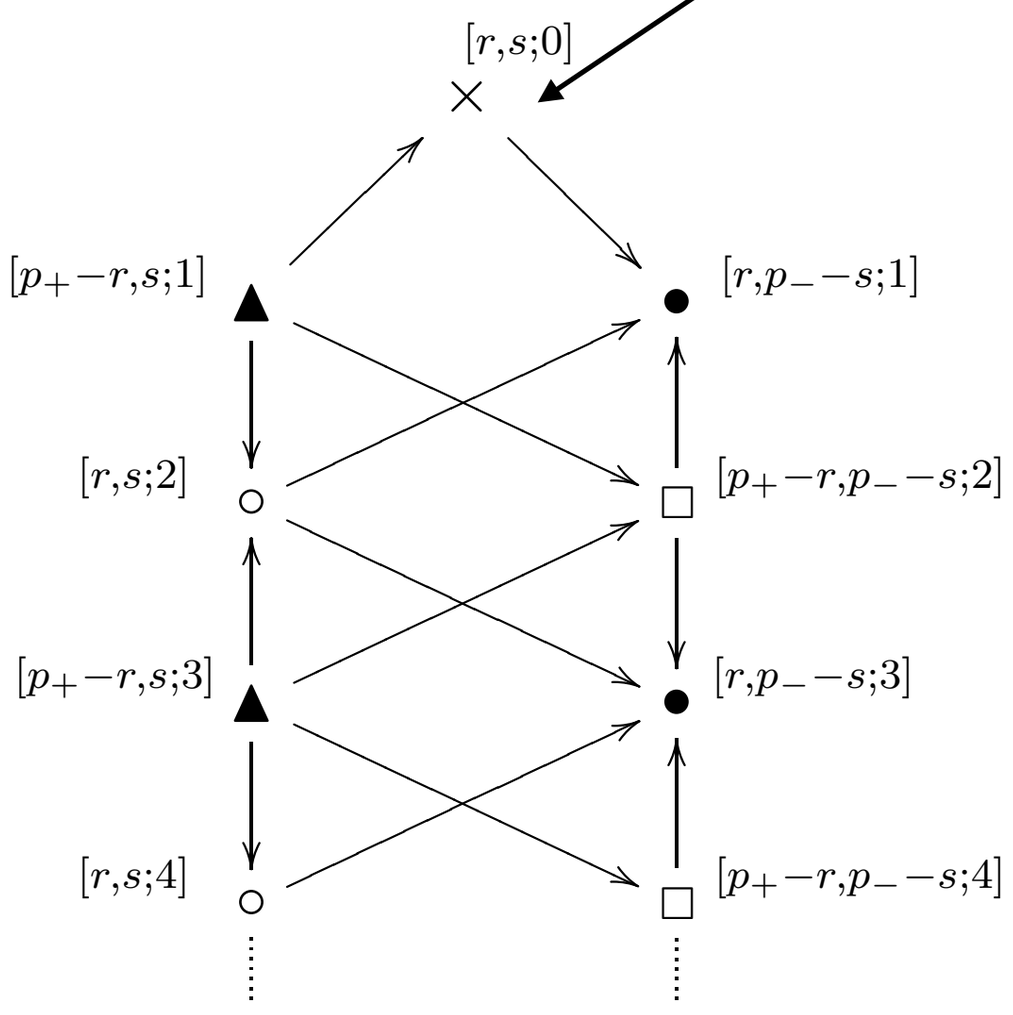
and constructed from the kernel

$$\bigcap_{i=1}^{N-1} \mathrm{Ker}\mathcal{Q}_{-}^{(i)[\nu_i]}$$

c.f. minimal models as VOA

irreducible Virasoro module $J_{r,s}$

[Feigin-Fuchs (1989)]



$$J_{r,s} = \mathrm{Ker} Q_+^{[r]} \cap Q_-^{[s]} / \mathrm{Im} Q_+^{[p_+-r]} \cap Q_-^{[p_--s]}$$

p_+=N, p_-=N+M

Summary

We investigated the Schur index with surface operator inserted.

1) in (A_{N-1}, A_{M-1}) theories with gcd(N,M)=1 and (A_1, A_{2k-3}) theories, we showed that the surface operators we considered reproduce all (and only) the characters of modules of the 2d chiral algebra.

2) in (A_{N-1}, $A_{N(k-1)-1}$) theories for N>2, surface operators give rise to a character formula conjectured to describe all non-vacuum modules of $\mathcal{B}(k)_{A_{N-1}}$.

Open problems and Future directions:

1. examination of OPE

   z dependence?

   c.f. fusion rule in Virasoro minimal model ☑

   How to derive h=-1/5 for s=1 in (A_1,A_2) theory?

2. more complicated models (for chiral algebra)

3. Macdonald and Hall-Littlewood limit?

   [Watanabe-RZ (WIP)]

4. chiral algebra even for non-conformal 4d theories?

5. (N=1) Lagrangian description for AD theories?

   [Maruyoshi-Song (2016, 2017)]

Chiral Algebra from 4d N=2 Superconformal Algebra

[Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees (2013)]

4d $\xrightarrow{\mathbb{R}^2 \times C}$ **2d N=(0,4) (scale invariance) symmetry on C**

$$L_{-1} = P_{++}, \quad L_{+1} = K^{++}, \quad 2L_0 = H + M,$$
$$\bar{L}_{-1} = P_{--}, \quad \bar{L}_{+1} = K^{--}, \quad 2\bar{L}_0 = H - M.$$

+ fermionic part

$$Q^I = Q^I_-, \quad \bar{Q}_I = \bar{Q}_{I\dot{-}}, \quad S_I = S^-_I, \quad \bar{S}^I = \bar{S}^{I\dot{-}}.$$

twist the symmetry with R-symmetry as

$$\hat{L}_{+1} = \bar{L}_{+1} - R^+, \quad \hat{L}_0 = \bar{L}_0 - R, \quad \hat{L}_- = \bar{L}_{-1} + R^-,$$

then for

$$\mathbb{Q}_1 := Q^1 + \bar{S}^2, \quad \mathbb{Q}_2 := S_1 - \bar{Q}_2,$$

the twisted global conformal part becomes Q-exact

$$\hat{L}_{-1} = \{\mathbb{Q}_1, \bar{Q}_1\} = \{\mathbb{Q}_2, -Q^2\}, \quad \hat{L}_{+1} = \{\mathbb{Q}_1, S_2\} = \{\mathbb{Q}_2, \bar{S}^1\}, \quad 2\hat{L}_0 = \{\mathbb{Q}_1, \mathbb{Q}_1^\dagger\} = \{\mathbb{Q}_2, \mathbb{Q}_2^\dagger\}.$$

A Possibly More Systematic Way to obtain the Chiral Algebra

(even work for non-SCFTs)

[Oh-Yagi (2019)]

[Jeong (2019)]

4d $\xrightarrow[\text{topological-holomorphic twist}]{\Sigma \times C}$ **2d N=(2,2) B-model on** Σ **and a holomorphic theory on** C

just a product of two surfaces

$\longrightarrow$ **chiral algebra on** C

Ω-deformation on Σ **(localize the d.o.f. on** Σ**)**

topological-holomorphic twist

$$M'_\Sigma = M_\Sigma + 2r\,,$$
$$M'_C = M_C + 2R\,,$$

which matches with the B-twist on Σ

cohomological supercharge $Q_K = \mathcal{Q}^1_- + \widetilde{\mathcal{Q}}_{2\dot{-}}$

and after the Ω-deformation

$$Q_K^\hbar = \mathcal{Q}^1_- + \widetilde{\mathcal{Q}}_{2\dot{-}} + 2\hbar(\widetilde{\mathcal{S}}^{2\dot{-}} - \mathcal{S}_1^-)\,,$$

whose cohomology is composed of those of $\mathbb{Q}_1$ **and** $\mathbb{Q}_2$ **.**

Macdonald index

- **The same set of BPS operators, i.e. Schur operators, contribute to the index.**

One therefore would like to interpret it as certain kind of refined character of the chiral algebra.

with refinement parameter T:=t/q

one of the best proposals: **[Song (2016)]**

$$\mathcal{I}(q,t) = \mathrm{tr}\left(T^{\#} q^{h}\right)$$

#: number of “basic” generators in the chiral algebra

“basic” generator: Virasoro operator in minimal models
current operators in affine KM algebra

ambiguity in this prescription:

the existence of null states, e.g.

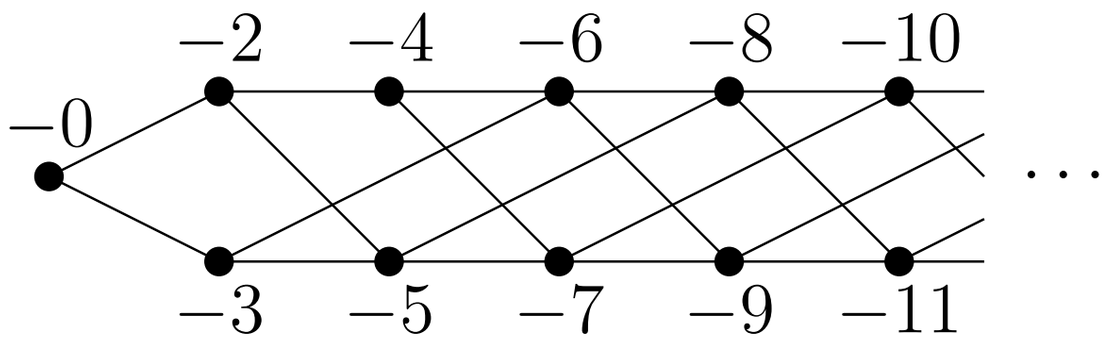
$$\left(-\frac{3}{5}L_{-4}+L_{-2}^2\right)\left|0,c=-\frac{22}{5}\right\rangle \ ,$$

Song's prescription: count the state with less #

There are also alternative ways to express the same quantity without ambiguity.

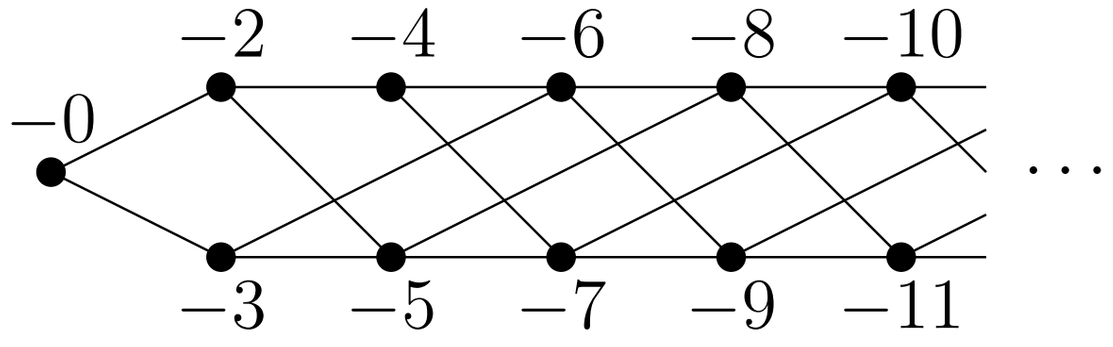
[Xie-Yan (2019)] [Watanabe-RZ (WIP)] [Foda (WIP)]…

- **Partially Ordered Sets (POSET)**



Virasoro character: counting the allowed orderings satisfying the partially order.
(The result contains a U(1) factor to be divided.)

Vertices labeled by {0,-2,-3,-4,-5,-6,-7,...}



The canonical ordering:

$$\pi(0) \geq \pi(-2) \geq \pi(-3) \geq \pi(-4) \geq \pi(-5) \geq \pi(-6) \geq \pi(-7) \geq \dots$$

whose character is given by

$$\chi(0, -2, -3, -4, -5, -6, -7, \dots) = \frac{1}{(q;q)_\infty},$$

= U(1) factor (free boson)

The following ordering, however,

$$\pi(0) \geq \pi(-3) > \pi(-2) \geq \pi(-4) \geq \pi(-5) \geq \pi(-6) \geq \pi(-7) \geq \dots$$

gives rise to the character

$$\chi(0, -3, -2, -4, -5, -6, -7, \dots) = \frac{q^2}{(q;q)_\infty}.$$

Realization of Macdonald index

 # = inverted pairs of neighboring vertices

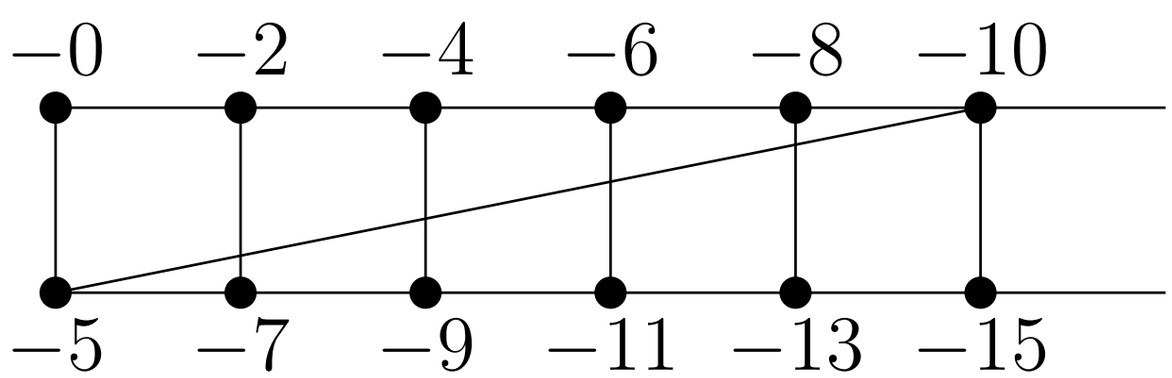
$$(0, -3_>, -2, -4, -5, -6, -7, -8, -9, -10, -11, \dots) \mapsto q^2 T,$$

Realization of Macdonald index

 # = inverted pairs of <u>neighboring</u> vertices

$$(0, -3_>, -2, -4, -5, -6, -7, -8, -9, -10, -11, \dots) \mapsto q^2 T,$$

e.g. (2,9) minimal model



at level 5,

$$(0, -2, -4, -5, -7_>, -6, -8, -9, -10, \dots) \mapsto t q^4,$$
$$(0, -2, -4, -6, -8_>, -5, -7, -9, -10, \dots) \mapsto t^2 q^3.$$

Bad news:

[Watanabe-RZ (WIP)]

- **Song's prescription does not seem to work when surface operator is inserted.**

  minimal models

  It (mysteriously) still works perfectly for s=0 (vacuum module) and s=1.

  for s=2, we need to compensate an additional strip-off factor

$$\frac{(1-q^{-2})(1-t^3)(1-t^4q^2)}{(1-t^2)(1-t^2q)(1-t^5q)} \times \cdots$$

  infinite product, and depends on the theory.

- **higher-rank case**

  How to assign #?

Summary II

3) We also speculated the correspondence rule between surface operators and modules via the screening charges.

$$\text{Ker } Q/\text{Im } Q$$

4) We tried to understand the same story in the Macdonald limit

In (A_1, A_{2k}) series, Song's proposal that Macdonald index is a refined character works for s=0 (trivial) and s=1 surface operators, but does not match with the Macdonald index unless we adopt a very complicated strip-off factor.

For those who are ready to decouple:

We could prove (rigorously):

there is a clear correspondence between the label of surface operators and the label of modules of chiral algebra, for

1) (A_{N-1}, A_{M-1}) theories with gcd(N,M)=1

and

2) (A_1, A_{2k-3}) theories

in particular, the Schur index with surface operator inserted reproduces the characters of all (and only) the modules in the dual chiral algebra.